Row and Column Space of a Matrix

The **row space** of a matrix $A \in \mathbb{R}^{m \times n}$ is the linear span of its rows (as n-vectors).

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Corollary

The dimension of the row space of $A \in \mathbb{R}^{m \times n}$ is the number of non-zero rows of the row echelon form of A.

The **column space** of a matrix $A \in \mathbb{R}^{m \times n}$ is the linear span of its columns (as m-vectors).

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If $A \in \mathbb{R}^{m \times n}$ and $C \in \mathbb{R}^{m \times m}$ is invertible, then the dimension of the column spaces of $C \cdot A$ is identical.

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The dimension of the column space of $A \in \mathbb{R}^{m \times n}$ is the number of non-zero rows of the row echelon form of A.

Summary: Let $A \in \mathbb{R}^{m \times n}$ be a matrix.

1. The row space is the linear span of A's rows. The column space is the linear span of A's columns.
2. Elementary row operations do not change the row space.
3. Elementary row operations do not change *the dimension* of the column space.
4. The dimensions of the row and column spaces of A are the same as for the reduced row echelon form A.

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What are the dimensions of row and column spaces?

$$\begin{pmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 4 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Corollary

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We call that dimension the **rank** of A.

A matrix $A \in \mathbb{R}^{n \times n}$ with rank n is called **full rank**.

Theorem

A matrix $A \in \mathbb{R}^{n \times n}$ is invertible if and only if it has full rank.

Proof.

For $A \in \mathbb{R}^{n \times n}$ the following statements are easily seen to be equivalent:

1. A is invertible.
2. The row echelon form of A is invertible.
3. The row echelon of A has n non-zero rows.
4. A has an n-dimensional row space.
5. A has full rank.

$\square$

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Questions?