MATH 109 – HOMEWORK 1

Due Friday 19th. Handwritten submissions only.

Exercise 1 (2 points)
We have seen in the lecture that the logical connective $\vee$ can be expressed in terms of $\neg$ and $\wedge$. It turns out that a single logical connective is sufficient to express all other logical connectives.

Let the logical connective $\bar{\wedge}$ be defined by

$$P \,\bar{\wedge}\, Q :\iff \neg(P \wedge Q).$$

Express the logical connectives $\wedge$, $\vee$, and $\neg$ in terms of the logical connective $\bar{\wedge}$.

Solution 1
We observe

$$\begin{aligned} \neg P &\iff P \,\bar{\wedge}\, P, \\ P \wedge Q &\iff \neg\left(P \,\bar{\wedge}\, Q\right) \iff \left(P \,\bar{\wedge}\, Q\right) \bar{\wedge} \left(P \,\bar{\wedge}\, Q\right), \\ P \vee Q &\iff \neg\left((\neg P) \wedge (\neg Q)\right) \\ &\iff (\neg P) \,\bar{\wedge}\, (\neg Q) \iff \left(P \,\bar{\wedge}\, P\right) \bar{\wedge} \left(Q \,\bar{\wedge}\, Q\right) \end{aligned}$$

Exercise 2 (2 points)
Describe all pairs (x, y) of real numbers x and y that satisfy the following equation:

$$x^2 + 2x + 1 = y^2 - 6y + 9.$$

Solution 2
Let (x, y) be a solution. We observe that

$$x^2 + 2x + 1 = y^2 - 6y + 9 \iff (x + 1)^2 = (y - 3)^2.$$

Hence

$$x + 1 = y - 3 \vee x + 1 = -y + 3 \vee -x - 1 = y - 3 \vee -x - 1 = -y + 3$$

However, we also observe that

$$x + 1 = y - 3 \iff -x - 1 = -y + 3, \quad x + 1 = -y + 3 \iff -x - 1 = y - 3.$$

Hence the law of absorption gives that (x, y) satisfy

$$x + 1 = y - 3 \vee x + 1 = -y + 3.$$

In other words,

$$x + 4 = y \vee -x + 2 = y.$$

All of the derivations so far have been equivalences, hence this last condition is necessary and sufficient. Geometrically, the solution set is the union of two lines.

Exercise 3 (2 points)
Recall the binomial coefficient for integer parameters $0 \leq k \leq n$. Prove that

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

Solution 3
For integers k and n with $0 \leq k \leq n$ we find

$$\begin{aligned}\binom{n}{k} + \binom{n}{k+1} &= \frac{(n)!}{(k)! \cdot (n-k)!} + \frac{(n)!}{(k+1)! \cdot (n-k-1)!} \\ &= \frac{(n)!(k+1)}{(k+1)! \cdot (n-k)!} + \frac{(n)!(n-k)}{(k+1)! \cdot (n-k)!} \\ &= \frac{(n+1)!}{(k+1)! \cdot (n-k)!} \\ &= \binom{n+1}{k+1}.\end{aligned}$$

Exercise 4 (2 points)
Given a real number g, find all real numbers x for which

$$\big||x-1| + x - 3\big| < g.$$

Solution 4
We note that the inequality has no solution if $g \leq 0$, so we assume that $g > 0$.

We conduct a case distinction. We have $|x-1| = x-1$ or $|x-1| = 1-x$, and we consider each of these cases separately.

First, assume that $|x-1| = x-1$. Then the original inequality holds if and only if

$$\big|x - 1 + x - 3\big| < g \iff \big|2x - 4\big| < g$$

We make another case distinction: we have $\big|2x-4\big| = 2x-4$ or $\big|2x-4\big| = 4-2x$. In the former case, we have $2x - 4 \geq 0$and the original inequality holds if and only if

$$2x - 4 < g,$$

and in the latter case, the original inequality holds if and only if

$$4 - 2x < g.$$

Hence, the original inequality holds if $x \geq 2$ and $2x - 4 < g$ or if $1 \leq x \leq 2$ and $4 - 2x < g$.

Second, assume that $|x-1| = -(x-1) = 1-x$. Then the original inequality holds if and only if

$$\big| -x + 1 + x - 3\big| < g \iff \big| -2\big| < g.$$

Hence if $|x-1| = 1-x$, then the original inequality holds if and only if $2 < g$.

We summarize this as follows: the original inequality holds if $x \geq 1$ and any of the two conditions $2x - 4 < g$ or $4 - 2x < g$ is true, or if $x < 1$ and $2 < g$.

Exercise 5 (2 points)
Prove the following: if x is an integer with at most three decimal digits $a_1 a_2 a_3$, then x is divisible by 3 if and only if $a_1 + a_2 + a_3$ is divisible by 3.

Solution 5
According to the problem statement, x is an integer and there exists integers a_1, a_2, a_3 between 0 and 9 such

that

$$
\begin{aligned}
x &= 100a_1 + 10a_2 + a_3 \\
&= (99+1)a_1 + (9+1)a_2 + a_3. \\
&= 99a_1 + 9a_2 + a_1 + a_2 + a_3.
\end{aligned}
$$

Hence we have

$$\frac{x}{3} = 33a_1 + 3a_2 + \frac{a_1 + a_2 + a_3}{3}.$$

We know that $33a_1 + 3a_2$ is an integer, since the sums and products of integers produces integers again.

Next, we observe that

$$\frac{x}{3} \in \mathbb{Z} \iff \frac{a_1 + a_2 + a_3}{3} \in \mathbb{Z}.$$

Indeed, if $(a_1 + a_2 + a_3)/3 \in \mathbb{Z}$, then

$$\frac{x}{3} = 33a_1 + 3a_2 + \frac{a_1 + a_2 + a_3}{3} \in \mathbb{Z},$$

and if $x/3 \in \mathbb{Z}$, then

$$\frac{a_1 + a_2 + a_3}{3} = \frac{x}{3} - 33a_1 - 3a_2 \in \mathbb{Z}.$$

Now, by definition of divisibility, $x/3$ is an integer if and only if x is divisible by 3, and $(a_1 + a_2 + a_3)/3$ is an integer if and only if $a_1 + a_2 + a_3$ is divisible by 3.

This had to be shown, and the proof is complete.

Exercise 6 (3 points)
A square number is an integer that is the square of another integer. Let x and y be two integers, each of which can be written as the sum of two square numbers. Show that the product xy can be written as the sum of two square numbers.

Solution 6
We assume that

$$x = a^2 + b^2, \quad y = c^2 + d^2,$$

for integers a, b, c, d. Hence

$$
\begin{aligned}
xy &= \left(a^2 + b^2\right)\left(c^2 + d^2\right) \\
&= (ac)^2 + (ad)^2 + (bc)^2 + (bd)^2 \\
&= (ac)^2 + 2(ac)(bd) + (bd)^2 + (ad)^2 - 2(ad)(bc) + (bc)^2 \\
&= (ac - bd)^2 + (ad - bc)^2
\end{aligned}
$$

Exercise 7 (3 points)
Let a and b be rational numbers. Consider the polynomial

$$p(x) = ax^2 + bx + (a + b).$$

Show that if $p(0)$ and $p(-1)$ are integers, then $p(x)$ is an integer for every integer x.

Solution 7
From the assumptions we get that

$$p(0) = a + b, \quad p(-1) = a - b + a + b = 2a$$

are integers. Since $2a$ is an integer, we have $a = \frac{1}{2}c$ for some integer c. But since $a + b$ is an integer, we have $c/2 + b$ being an integer. Hence there exists an integer d such that $b = d/2$.

Thus we can write

$$p(x) = \frac{c}{2}x^2 + \frac{d}{2}x + \frac{c+d}{2}.$$

Moreover, since $(c + d)/2$ is an integer, we get that c and d are either both odd or both even.

If c and d are both even, then $p(x)$ clearly is an integer for all integers x. If instead c and d are both odd, then for all even integers x the value $p(x)$ is easily seen to be an integer, and for all odd integers x the sum $cx^2 + dx$ is the sum of two odd numbers and hence even. It follows that $p(x)$ is integer in that case too.