MATH 109 – HOMEWORK 3

Due Friday, February 2nd. Handwritten submissions only.
The exercises in this homework are worth 16 points.

Exercise 1
Let A, B, and C be sets. Prove the following statements:

- $(A \setminus B) \setminus C = A \setminus (B \cup C)$
- $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$
- $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$
- $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$
- $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

Hint: Use the correspondence between elementary set operations and logical connectives.

Solution 1
We prove each of these statements:

- If from a set A we remove everything in B and everything in C, then this is the same as removing from A the union of B and C. Formally,

$$\begin{aligned} x \in (A \setminus B) \setminus C &\iff x \in (A \setminus B) \wedge x \notin C \\ &\iff x \in A \wedge x \notin B \wedge x \notin C \\ &\iff x \in A \wedge \neg (x \in B \vee x \in C) \\ &\iff x \in A \wedge \neg (x \in B \cup C) \\ &\iff x \in A \wedge x \notin B \cup C \\ &\iff x \in A \setminus (B \cup C) \end{aligned}$$

- If from a set A we remove everything in B except everything in C, then this is the same as removing B from A and adding the intersection of A and C again. Formally,

$$\begin{aligned} x \in (A \setminus (B \setminus C)) &\iff x \in A \wedge x \notin B \setminus C \\ &\iff x \in A \wedge \neg (x \in B \setminus C) \\ &\iff x \in A \wedge \neg (x \in B \wedge x \notin C) \\ &\iff x \in A \wedge (x \notin B \vee x \in C) \\ &\iff (x \in A \wedge x \notin B) \vee (x \in A \wedge x \in C) \\ &\iff (x \in A \setminus B) \vee (x \in A \cap C) \\ &\iff x \in (A \setminus B) \cup (A \cap C) \end{aligned}$$

- Formally,

$$\begin{aligned} x \in (A \cap B) \setminus C &\iff x \in A \cap B \wedge x \notin C \\ &\iff x \in A \wedge x \in B \wedge x \notin C \\ &\iff x \in A \wedge x \notin C \wedge x \in B \wedge x \notin C \\ &\iff x \in A \setminus C \wedge x \in B \setminus C \\ &\iff x \in (A \setminus C) \cap (B \setminus C) \end{aligned}$$

- Formally,

$$\begin{aligned} x \in (A \cup B) \setminus C &\iff x \in A \cup B \wedge x \notin C \\ &\iff (x \in A \vee x \in B) \wedge x \notin C \\ &\iff (x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C) \\ &\iff x \in A \setminus C \vee x \in B \setminus C \\ &\iff x \in (A \setminus C) \cup (B \setminus C) \end{aligned}$$

- Formally,

$$\begin{aligned} x \in A \setminus (B \cap C) &\iff x \in A \wedge x \notin B \cap C \\ &\iff x \in A \wedge \neg (x \in B \cap C) \\ &\iff x \in A \wedge \neg (x \in B \wedge x \in C) \\ &\iff x \in A \wedge (x \notin B \vee x \notin C) \\ &\iff (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C) \\ &\iff (x \in A \setminus B) \vee (x \in A \setminus C) \\ &\iff x \in (A \setminus B) \cup (A \setminus C) \end{aligned}$$

- Formally,

$$\begin{aligned} x \in A \setminus (B \cup C) &\iff x \in A \wedge x \notin B \cup C \\ &\iff x \in A \wedge \neg (x \in B \cup C) \\ &\iff x \in A \wedge \neg (x \in B \vee x \in C) \\ &\iff x \in A \wedge (x \notin B \wedge x \notin C) \\ &\iff (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C) \\ &\iff (x \in A \setminus B) \wedge (x \in A \setminus C) \\ &\iff x \in (A \setminus B) \cap (A \setminus C) \end{aligned}$$

Exercise 2
For each of the sums

$$A_n := \sum_{k=1}^{n} k, \quad B_n := \sum_{k=1}^{n} k^5,$$

determine the natural numbers $n \in \mathbb{N}$ for which the respective sum is even. Prove your result.

Solution 2
We may recall that

$$A_n = \frac{(n+1)n}{2}.$$

We let $r \in \{0, 1, 2, 3\}$ be the remainder of n after divison by 4, hence there exists $k \in \mathbb{N}_0$ such that $n = 4q + r$. This leads to

$$A_n = \frac{(4q + r + 1)(4q + r)}{2} = \frac{16q^2 + 12qr + (r + r^2)}{2}.$$

The terms $8q^2$ and $6qr$ are always even. Hence A_n is even if and only if $(r + r^2)/2$ is even. This is the case if $r = 0$ or $r = 1$ or $r = 3$, but not if $r = 2$.

We recall that whether A_n is even or odd depends only on the number of even and odd natural numbers smaller than or requal to n. Hence A_n is even if and only if B_n is even.

Solution 3

We see that x and y are positive integers, and hence z is a positive integer.

The trick is that the quotient of successive cubic numbers approaches 1 as the cubic numbers get bigger.

Exercise 3

Let x and y be real numbers. Prove the two inequalities

$$|x+y| \leq |x| + |y|, \quad |x-y| \geq \big||x| - |y|\big|.$$

Solution 4

Let $x, y \in \mathbb{R}$. Note that $x \leq |x|$ and $y \leq |y|$ is always true.

We prove the first inequality by a case distinction. If $|x+y| = x+y$, then we see

$$|x+y| = x+y \leq |x| + |y|.$$

If $|x+y| = -x-y$, then we see

$$|x+y| = (-x) + (-y) \leq |x| + |y|.$$

Hence the first inequality is always true.

We prove the second inequality using the first inequality. We have

$$|x| = |x-y+y| \leq |x-y| + |y|,$$

which gives

$$|x| - |y| \leq |x-y|$$

Similarly, we have

$$|y| = |y-x+x| \leq |y-x| + |x| \leq |x-y| + |x|,$$

which gives

$$|y| - |x| \leq |x-y|$$

In summary, we observe

$$|x-y| \geq \max\{|y| - |x|, |x| - |y|\} = \big||x| - |y|\big|,$$

which had to be proven.

Exercise 4

Let A, B, C, and D be sets with ten elements each, and suppose that the intersections of two of each have at least nine elements.

(1) Show that the intersection $A \cap B \cap C$ is non-empty.
(2) Show that the intersection $A \cap B \cap C \cap D$ is non-empty.

Solution 5

Let A, B, C, and D be as in the statement.

Suppose that $A \cap B$ has exactly nine elements. Then there exists an element $a \in A$ such that $a \notin B$ and there exists an element $b \in B$ such that $b \notin A$.

(1) Consider now the intersection $A \cap B \cap C$. We make a case distinction: If $A = B = C$, then this intersection is not empty. If $A = B$ but $A \neq C$, then the reasoning above shows that there exists $a \in A$ such that

$$A \cap C = A \setminus \{a\} \neq \emptyset.$$

Finally, if A, B, C are all distinct to each other, then we first observe that $A \cap B = A \setminus \{a\}$ for some $a \in A$. On the other hand, $A \cap C = A \setminus \{a'\}$ for some $a' \in A$. Hence,

$$A \cap B \cap C = (A \cap B) \cap (A \cap C) = (A \setminus \{a\}) \cap (A \setminus \{a'\}) = A \setminus \{a, a'\}$$

Since A has ten elements, we see that $A \cap B \cap C$ has at least eight elements.

(2) Show that the intersection $A \cap B \cap C \cap D$ is non-empty. If these four sets are not pairwise distinct, then we can reuse the previous results. If they are distinct, then we first recall

$$A \cap B \cap C = A \setminus \{a, a'\}$$

for some $a, a' \in A$. Now we also have that $A \cap C = A \setminus \{a''\}$ for some $a'' \in A$. Hence

$$A \cap B \cap C \cap D = (A \cap B \cap C) \cap (A \cap D) = (A \setminus \{a, a'\}) \cap (A \setminus \{a''\}) = (A \setminus \{a, a', a''\}).$$

Hence the intersection has at least 7 elements.