# MATH 109 – HOMEWORK 5

*Due Friday, February 16th. Handwritten submissions only.*
*The exercises in this homework are worth 16 points.*

**Exercise 1**
Let $a, b \in \mathbb{N}_0$ with $b \neq 0$. Prove that

$$\gcd(a,b) = \gcd\left(b, a - \left\lfloor \frac{a}{b} \right\rfloor b\right).$$

Here, $\lfloor q \rfloor$ denotes the largest integer not larger than $q \in \mathbb{R}$.
*Hint: Use a similarly looking result seen previously in the lecture.*

**Exercise 2**
Let $a, b \in \mathbb{N}_0$ such that $a > b$ and let $q, r \in \mathbb{N}_0$ with

$$a = q \cdot b + r, \quad 0 \leq r < d.$$

Show that $r < a/2$.

**Exercise 3**
Prove the following statement on the run-time of the Euclidean algorithm: if $a, b \in \mathbb{N}$ and $k \in \mathbb{N}_0$ such that $a < 2^k$ and $b < 2^k$, then the Euclidean algorithm takes at most $2k$ steps.

*Remark: This statement shows that the Euclidean algorithm has a run-time that growth at most logarithmically in the number of digits of the two input variables.*

**Exercise 4**
Prove the correctness of the naive method to find the greatest common divisor of two numbers. You may restrict yourself to the special case that $a \neq 0$ and $b \neq 0$.

**Exercise 5**
Prove the following statement: if $A \subseteq \mathbb{N}_0$ and $c \in \mathbb{N}_0$ such that $a \leq c$ for all $a \in A$, then there exists $m \in A$ such that $a \leq m$ for all $a \in A$.

**Exercise 6**
We define a recursive series $a_1, a_2, a_3 \dots$ as follows. We let $a_1 = a$ for some $a \in \mathbb{N}$ and for all $k \in \mathbb{N}$ we define

$$a_{k+1} = \begin{cases} a_k/2 & \text{if } a_k \text{ is even,} \\ a_k + 1 & \text{if } a_k \text{ is odd.} \end{cases}$$

Show that for every choice of $a$ there exists $k \in \mathbb{N}$ such that $a_k = 1$.

*Hint: to get an idea of what is going on, pick some $a \in \mathbb{N}$ and write down $a_1, a_2, a_3$. For example, try $a = 11, 56, 1025$.*

**Exercise 7**

We define a recursive series $a_1, a_2, a_3 \ldots$ as follows. We let $a_1 = a$ for some $a \in \mathbb{N}$ and for all $k \in \mathbb{N}$ we define

$$a_{k+1} = \begin{cases} a_k/2 & \text{if } a_k \text{ is even,} \\ 3a_k + 1 & \text{if } a_k \text{ is odd.} \end{cases}$$

Show that for every choice of $a$ there exists $k \in \mathbb{N}$ such that $a_k = 1$. *Hint: this is the Collatz conjecture and generally considered as one of the hardest open problems in mathematics. Don't try to solve it! If you happen to solve it, you will pass this course with A+ and receive a Fields medal.*