Part 2: Set Theory

New Sets From Old

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Power sets

The **power set** $\mathfrak{P}(A)$ of a set A is the set of all subsets of A. In other words, for every set S we have

$$S \in \mathfrak{P}(A) \quad :\Longleftrightarrow \quad S \subseteq A.$$

We note that always

$$\emptyset, A \in \mathfrak{P}(A).$$

The construction of the power set can be repeated:

$$\mathfrak{P}(\mathfrak{P}(A)), \quad \mathfrak{P}(\mathfrak{P}(\mathfrak{P}(A))), \quad \dots$$

The existence of the power set is an axiom of set theory.

Example for a power set

Let $A = \{0, 1\}$. Then

$$\mathfrak{P}(A) = \big\{\emptyset, \{0\}, \{1\}, \{0, 1\}\big\}.$$

Let $B = \{\mathbb{N}, \mathbb{Q}, \mathbb{R}\}$. Then

$$\mathfrak{P}(A) = \big\{\emptyset, \{\mathbb{N}\}, \{\mathbb{Q}\}, \{\mathbb{R}\}, \{\mathbb{Q}, \mathbb{R}\}, \{\mathbb{N}, \mathbb{R}\}, \{\mathbb{N}, \mathbb{Q}\}, \{\mathbb{N}, \mathbb{Q}, \mathbb{R}\}\big\}.$$

Let $C = \{3, 5, 7\}$. Then

$$\mathfrak{P}(A) = \big\{\emptyset, \{3\}, \{5\}, \{7\}, \{5, 7\}, \{3, 7\}, \{3, 5\}, \{3, 5, 7\}\big\}.$$

Examples of power set

Consider the power set of the empty set. We have

$$\mathfrak{P}(\emptyset) = \{\emptyset\}.$$

$$\mathfrak{P}(\mathfrak{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}.$$

$$\mathfrak{P}(\mathfrak{P}(\emptyset)) = \big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\big\}.$$

Exercise: Given a set with n elements, how many element does the power set contain?

Set Union, Set Intersection, and Set Difference

Let A and B be two sets. We define their **union** $A \cup B$ by

$$x \in A \cup B \quad :\Longleftrightarrow \quad x \in A \vee x \in B.$$

We define their **intersection** $A \cap B$ by

$$x \in A \cap B \quad :\Longleftrightarrow \quad x \in A \wedge x \in B.$$

Finally, we define their **difference** by

$$A \setminus B \quad :\Longleftrightarrow \quad x \in A \wedge x \notin B..$$

Remarks:

- The existence of the union of two sets is not trivial. It is an axiom of set theory.
- The intersection of two sets can be defined via set comprehension

$$A \cap B := \{ x \in A \mid x \in B \}$$

- The difference of two sets can be defined via set comprehension

$$A \setminus B := \{ x \in A \mid x \notin B \}.$$

Commutative Law:

$$A \cap B = B \cap A,$$
$$A \cup B = B \cup A.$$

Associate Law:

$$A \cap (B \cap C) = (A \cap B) \cap C,$$
$$A \cup (B \cup C) = (A \cup B) \cup C.$$

Distributive Law:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Idempotency Law:

$$A \cap A = A,$$
$$A \cup A = A.$$

Absorption Law:

$$A \cap (A \cup B) = A,$$
$$A \cup (A \cap B) = A.$$

Laws concerning the empty set:

$$A \cup \emptyset = A, \quad A \cap \emptyset = \emptyset, \quad A \setminus \emptyset = A.$$

We can use the logical calculus to discuss set theory.

$$A \cap B = B \cap A.$$

Proof: For every x we have

$$\begin{aligned} x \in A \cap B &\iff x \in A \wedge x \in B \\ &\iff x \in B \wedge x \in A \iff x \in B \cap A. \end{aligned}$$

We can use the logical calculus to discuss set theory.

$$A \setminus (B \cap A) = A \setminus B$$

Proof: We have for every x that

$$\begin{aligned}
x \in A \setminus (A \cap B) &\iff x \in A \wedge x \notin A \cap B \\
&\iff x \in A \wedge \neg(x \in A \cap B) \\
&\iff x \in A \wedge \neg(x \in A \wedge x \in B) \\
&\iff x \in A \wedge (\neg(x \in A) \vee \neg(x \in B)) \\
&\iff x \in A \wedge (x \notin A \vee x \notin B) \\
&\iff (x \in A \wedge x \notin A) \vee (x \in A \wedge x \notin B) \\
&\iff F \vee (x \in A \wedge x \notin B) \\
&\iff x \in A \wedge x \notin B \\
&\iff x \in A \setminus B
\end{aligned}$$